HIMACHAL PRADESH UNIVERSITY-2021

SOLVED PAPER

B.A./B.Sc.-III (Annual) MATHEMATICS

(MATRICES)

Time : 3 Hours **Maximum Marks : 70**

Note : Attempt *five* questions in all. Section A (Question No. 1) is compulsory. Attempt four questions from Section-B, selecting one question each from the Units-I, II, III and IV. Marks are given against questions.

SECTION – A

LBSNAA

Compulsory Question

1. (*i*) Define orthogonal and Unitary matrix.

Sol. Unitary Matrix : A square matrix P over the field of complex numbers is said to be unitary if and only if $P^{\theta} P = I$.

SHIVANG GARG

Orthogonal Matrix : A square matrix P over the field of reals is said to be orthogonal if and only if $P'P = I$.

(*ii*) Define consistent and inconsistent system of linear equations.

Sol. Consistent System : A system of equations is said to be consistent if its solution (one or more) exists.

Inconsistent System : A system of equations is said to be inconsistent if its solution does not exist.

(*iii*) Define rank of a matrix.

Sol. Rank of a Matrix : A number r is said to be rank of a non-zero matrix A if

(*a*) there exists at least one minor of order r of A which does not vanish, and

(*b*) every minor of order $(r + 1)$, if any, vanishes

(*iv*) Determine the rank of matrix : $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & 4 & 2 \end{bmatrix}$.

Sol. $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & 4 & 2 \end{bmatrix}$

Now $\begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} = -1 - 0 = -1$, which is not zero.

$\therefore$ A possesses a non-zero minor of order 2.

$\therefore \rho(A) \geq 2.$ ...(1)

$\because$ A does not possess any minor of order 3.

$\therefore \rho(A) \le 2$ ...(2)

From (1) and (2), we get,

$$\rho(A) = 2.$$

(*v*) Define eigen-value and eigen-vectors of a matrix.

Sol. Eigen Values and Eigen Vectors of a Matrix

Let A be a square matrix of order n over a field F. A scalar $\lambda \in F$ is called an **eigen value** of A iff there exists a non-zero $n \times 1$ column matrix

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ such that } AX = \lambda X$$

The non-zero column matrix X is called the **eigen vector** of the matrix A corresponding to the eigen value λ of A.

(*vi*) Define basis of a vector space.

Sol. Basis : Let V (F) be a vector space. A subset B of V is called a basis of V iff

(*a*) B is linearly Independent set

(*b*) L (B) = V *i.e.,* B generates (spans) V.

or in other words every element in V is a linear combination of the elements of B.

(*vii*) Define Dilation Mapping.

Sol. Dilation : A mapping $T : R^n \to R^m$ is known as a dilation or magnification if $T(X) = kX$ for some $k \in R^m$ is a fixed constant and $X \in R^n$ is any vector and $k > 1$.

But if $0 < k < 1$, then this mapping is known as a contraction.

(*viii*) Show that the transformation matrix T is a pure rotation, where

$$T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

Sol. We know that when the direction of axis without changing the origin, is changed then relation between old coordinates (x, y) and new coordinates (x', y') is given by

$$x' = x\cos\theta + y\sin\theta$$

$$y' = -x\sin\theta + y\cos\theta$$

or
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$\therefore$ transformation T is given by

$$T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

(2 × 8 = 16)

SECTION – B

Unit-I

2. (*a*) If A is a skew-symmetric matrix and I + A is non-singular, then show that $B = (I - A)(I + A)^{-1}$ is orthogonal matrix.

Sol. Since A is skew-symmetric.

$$\therefore \quad A' = -A \qquad \ldots(1)$$

Also I + A is non-singular $\Rightarrow (I + A)^{-1}$ exists.

Now $\quad B = (I - A)(I + A)^{-1}$ will be orthogonal

if $\quad B'B = I$

i.e. if $\quad [(I - A)(I + A)^{-1}]' [(I - A)(I + A)^{-1}] = I$

i.e. if $\quad [(I + A)^{-1}]' (I - A)' (I - A)(I + A)^{-1} = I$

i.e. if $\quad [(I + A)']^{-1} (I - A)' (I - A)(I + A)^{-1} = I$

i.e. if $\quad (I - A)^{-1} (I + A)(I - A)(I + A)^{-1} = I \qquad [\because \text{ of } (1)]$

i.e. if $\quad [(I - A)^{-1}(I - A)][(I + A)(I + A)^{-1}] = I$

i.e. if $\quad I \cdot I = I$

i.e. if $\quad I = I$, which is true.

Hence the result.

(*b*) Using elementary transformation, find the rank of the matrix

$$A = \begin{bmatrix} 1 & -1 & 1 & 5 \\ 2 & 1 & -1 & -2 \\ 3 & -1 & -1 & 7 \end{bmatrix}$$

(6½, 7)

Sol.
$$A = \begin{bmatrix} 1 & -1 & 1 & 5 \\ 2 & 1 & -1 & -2 \\ 3 & -1 & -1 & 7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 5 \\ 0 & 3 & -3 & -12 \\ 0 & 2 & -4 & -8 \end{bmatrix}, \text{ by } R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 3R_1$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 5 \\ 0 & 1 & 1 & -4 \\ 0 & 2 & -4 & -8 \end{bmatrix}, \text{ by } R_2 \to R_2 - R_3$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 5 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & -6 & 0 \end{bmatrix}, \text{ by } R_3 \to R_3 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & -6 & 0 \end{bmatrix}, \text{ by } C_2 \to C_2 + C_1, C_3 \to C_3 - C_1, C_4 \to C_4 - 5C_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -6 & 0 \end{bmatrix}, \text{ by } C_3 \to C_3 - C_2, C_4 \to C_4 + 4C_2$$

The rank of $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -6 & 0 \end{bmatrix}$ is 3 as the minor $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -6 \end{vmatrix} = -6 \neq 0$ of order 3 does not vanish

$\therefore \qquad \rho(A) = 3.$

3. (*a*) Reduce the matrix $\begin{bmatrix} 1 & -1 & 2 & -1 \\ 4 & 2 & -1 & 2 \\ 2 & 2 & -2 & 0 \end{bmatrix}$ to normal form. Hence find rank.

Sol. Let $A = \begin{bmatrix} 1 & -1 & 2 & -1 \\ 4 & 2 & -1 & 2 \\ 2 & 2 & -2 & 0 \end{bmatrix}$

$$\therefore \begin{bmatrix} 1 & -1 & 2 & -1 \\ 4 & 2 & -1 & 2 \\ 2 & 2 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 2 & -1 \\ 0 & 6 & -9 & 6 \\ 0 & 4 & -6 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

by $R_2 \to R_2 - 4R_1, R_3 \to R_3 - 2R_1$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & -9 & 6 \\ 0 & 4 & -6 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

by $C_2 \to C_2 + C_1, C_3 \to C_3 - 2C_1, C_4 \to C_4 + C_1$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{3}{2} & 1 \\ 0 & 4 & -6 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & \frac{1}{6} & 0 \\ -2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ by } R_2 \to \frac{1}{6} R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{3}{2} & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & \frac{1}{6} & 0 \\ \frac{2}{3} & -\frac{2}{3} & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ by } R_3 \to R_3 - 4R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & \frac{1}{6} & 0 \\ \frac{2}{3} & -\frac{2}{3} & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{3}{2} & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\text{by } C_3 \to C_3 + \frac{3}{2} C_2,\ C_4 \to C_4 - C_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & \frac{1}{6} & 0 \\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{2} \end{bmatrix} A \begin{bmatrix} 1 & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{3}{2} & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ by } R_3 \to -\frac{1}{2} R_3$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & \frac{1}{6} & 0 \\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{2} \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -1 & \frac{3}{2} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \text{ by } C_3 \leftrightarrow C_4$$

$$\Rightarrow [I_3 \ \ O] = PAQ$$

$$\text{where } P = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & \frac{1}{6} & 0 \\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{2} \end{bmatrix},\ Q = \begin{bmatrix} 1 & 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -1 & \frac{3}{2} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\therefore$ rank of A = 3.

(*b*) Show that $k = 6$ is the only real value for which the following equations have non-zero solution :

$$x + 2y + 3z = kx$$

$$3x + y + 2z = ky$$

$$2x + 3y + z = kz$$

(6½, 7)

Sol. The given equations are

$$x+2y+3z=kx$$
$$3x+y+2z=ky$$
$$2x+3y+z=kz$$

which can be written as

$$(1-k)x+2y+3z=0$$
$$3x+(1-k)y+2z=0$$
$$2x+3y+(1-k)z=0$$

or $$\begin{bmatrix}1-k & 2 & 3\\ 3 & 1-k & 2\\ 2 & 3 & 1-k\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=O$$

The equation has a non-zero solution

if $$\begin{vmatrix}1-k & 2 & 3\\ 3 & 1-k & 2\\ 2 & 3 & 1-k\end{vmatrix}=0$$

i.e. if $(1-k)\begin{vmatrix}1-k & 2\\ 3 & 1-k\end{vmatrix}-2\begin{vmatrix}3 & 2\\ 2 & 1-k\end{vmatrix}+3\begin{vmatrix}3 & 1-k\\ 2 & 3\end{vmatrix}=0$

i.e. if $(1-k)[(1-k)^2-6]-2[3(1-k)-4]+[9-2(1-k)]=0$

i.e. if $(1-k)(k^2-2k-5)-2(3-3k-4)+3(9-2+2k)=0$

i.e. if $-k^3+3k^2+3k-5+6k+2+21+6k=0$

i.e., if $k^3-3k^2-15k-18=0$

$k=6$ is a root of this equation $[\because 216-108-90-18=0]$

Remaining roots of this equation are given by

$$k^2+3k+3=0$$

$$\therefore \quad k=\frac{-3+i\sqrt{3}}{2}$$

$$\therefore \quad k=6,\frac{-3+i\sqrt{3}}{2},\frac{-3-i\sqrt{3}}{2}$$

$\therefore$ only real value of k is 6.

Unit-~~II~~ I

4. (*a*) Find A^{-1} by using row elementary operations, if $A=\begin{bmatrix}1 & 2 & 1\\ 3 & 1 & 2\\ 0 & 1 & 2\end{bmatrix}$

Sol. Let $A=\begin{bmatrix}1 & 2 & 1\\ 3 & 1 & 2\\ 0 & 1 & 2\end{bmatrix}$

Now $\quad A = IA$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & -1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A, \text{ by } R_2 \rightarrow R_2 - 3R_1$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 11 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix} A, \text{ by } R_2 \rightarrow R_2 + 6R_3$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -21 \\ 0 & 1 & 11 \\ 0 & 0 & -9 \end{bmatrix} = \begin{bmatrix} 7 & -2 & -12 \\ -3 & 1 & 6 \\ 3 & -1 & -5 \end{bmatrix} A, \text{ by } R_1 \rightarrow R_1 - 2R_2,\ R_3 \rightarrow R_3 - R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -21 \\ 0 & 1 & 11 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -2 & -12 \\ -3 & 1 & 6 \\ -\frac{1}{3} & \frac{1}{9} & \frac{5}{9} \end{bmatrix} A, \text{ by } R_3 \rightarrow -\frac{1}{9} R_3$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & -\frac{2}{9} & -\frac{1}{9} \\ -\frac{1}{3} & \frac{1}{9} & \frac{5}{9} \end{bmatrix} A, \text{ by } R_1 \rightarrow R_1 + 21R_3,\ R_2 \rightarrow R_2 - 11R_3$$

$$\Rightarrow \qquad I = A^{-1} A$$

$$\therefore \quad A^{-1} = \begin{bmatrix} 0 & \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & -\frac{2}{9} & -\frac{1}{9} \\ -\frac{1}{3} & \frac{1}{9} & \frac{5}{9} \end{bmatrix}$$

(*b*) Diagonalize the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$. (6½, 7)

Sol. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \Rightarrow \lambda I = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$\therefore \quad A - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{bmatrix}$

$\therefore$ Characteristic equation of matrix A is $|A - \lambda I| = 0$

or $\begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = 0$

or $(1-\lambda)(2-\lambda) - (3)(2) = 0$ or $\lambda^2 - 3\lambda + 2 - 6 = 0$

or $\lambda^2 - 3\lambda - 4 = 0$ or $(\lambda + 1)(\lambda - 4) = 0$

$\therefore$ $\lambda = -1, 4$ are the eigen values of A.

The egien vector $X = \begin{bmatrix} x \\ y \end{bmatrix} \neq O$ corresponding to the eigen value $\lambda = -1$ is given by

$AX = \lambda X$ or $AX = -X$

or $(A + I)X = O$

$\therefore \quad \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Now the coefficient matrix of thee equations is of rank 1. Therefore this equation has only $2 - 1 = 1$ L.I. solution. Thus there is only one L.I. eigen vector corresponding to the eigen value -1. These equations can be written as

$2x + 2y = 0$, or $y = -x$

Take $y = -1$, $\therefore$ $x = 1$

$X = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigen vector of A corresponding to the eigen value -1.

The eigen vector $X = \begin{bmatrix} x \\ y \end{bmatrix} \neq O$ corresponding to the eigen value $\lambda = 4$ is given by

$AX = 4X$, or $(A - 4I)X = O$

$\therefore \quad \begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -3 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore \quad -3x + 2y = 0$, or $x = \frac{2}{3} y$

Take $y = 3$, $\therefore$ $x = 2$

$\therefore \quad X = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is an eigen vector of A corresponding to the eigen value 4.

$\therefore \quad P = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$

$$|P| = \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} = 3 + 2 = 5$$

$$\text{adj. } P = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}' = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$$

$$P^{-1} = \frac{\text{adj.} P}{|P|} = \frac{1}{5}\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$$

$$P^{-1}AP = \frac{1}{5}\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3-6 & 6-4 \\ 1+3 & 2+2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} -3 & 2 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -3-2 & -6+6 \\ 4-4 & 8+12 \end{bmatrix} = \begin{bmatrix} -5 & 0 \\ 0 & 20 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$$, which is a diagonal matrix.

5. (*a*) Find the invariant points of the transformations defined by

$$x' = 1 - 2y,$$
$$y' = 2x - 3$$

Sol. For invariant points, we must have $x' = x$ and $y' = y$.

Therefore, $x' = 1 - 2y \Rightarrow x = 1 - 2y$

and $y' = 2x - 3 \Rightarrow y = 2x - 3$

$\therefore \quad x = 1 - 2(2x - 3) \Rightarrow x = 1 - 4x + 6 \Rightarrow 5x = 7 \Rightarrow x = \frac{7}{5}$

$\therefore \quad y = 2x - 3 = 2\left(\frac{7}{5}\right) - 3 = -\frac{1}{5}$

$\therefore \quad$ the invariant point is $(x, y) = \left(\frac{7}{5}, -\frac{1}{5}\right)$.

(*b*) Show that the following equations are consistent. Hence find the solution :

$$x - y + z = 5$$
$$2x + y - z = -2$$
$$3x - y - z = -7$$

(6½, 7)

Sol. The given equations are

$$x - y + z = 5$$
$$2x + y - z = -2$$
$$3x - y - z = -7$$

These equations can be written as

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 3 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ -7 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -3 \\ 0 & 2 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ -22 \end{bmatrix}, \text{ by } R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 3R_1$$

$$\therefore \quad \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ -22 \end{bmatrix}, \text{ by } R_2 \to \frac{1}{3}R_2,\ R_3 \to \frac{1}{2}R_3$$

$$\therefore \quad \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}, \text{ by } R_3 \to R_3 - R_2$$

Now rank of $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$ as well as of $\begin{bmatrix} 1 & -1 & 1 & 5 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & -1 & -7 \end{bmatrix}$ is 3

$\therefore$ given system of equation is consistent and has a unique solution which is given by

$$x - y + z = 5$$
$$y - z = -4$$
$$-z = -7 \quad \text{or} \quad z = 7$$

$\therefore \quad y - 7 = -4 \Rightarrow y = 3$

$\therefore \quad x - 3 + 7 = 5 \Rightarrow x = 1$

$\therefore$ solution is $x = 1,\ y = 3,\ z = 7.$

Unit-III

6. (*a*) Prove that a non empty subset W of a vector space V (F) is a subspace of V iff W is closed under addition and scalar multiplication.

Sol. Given W is a subspace of V (F)

$\therefore$ By def. of vector subspace, W is closed under addition and scalar multiplication.

Hence the result holds.

Conversely. It is given that W is closed under addition and scalar multiplication

$\Rightarrow$ For all $x, y \in W, \alpha \in F$

we have $x + y \in W$ ...(1)

$\alpha x \in W$ ...(2)

We have to prove W is a subspace of V (F).

Now $-1 \in F$ and $x \in W$

$\Rightarrow (-1)\, x \in W$ (Using 2)

$\Rightarrow -x \in W$

[Since $x \in W$ implies $x \in V$ and $(-1)\,x = -x$ holds in V]

so that addative inverse of every element in W exists.

And $\forall\ x \in W,\ -x \in W$

$\Rightarrow x + (-x) \in W$ (Using 1)

$\Rightarrow \quad \mathbf{0} \in W$

so that addative identity exists in W.

$\therefore\ x + \mathbf{0} = x = \mathbf{0} + x\ \forall\ x \in W$

and $\forall\ x \in W,$ there exists $-x \in W$ such that

$$x + (-x) = \mathbf{0} = (-x) + x.$$

Now, since $W \subset V$ *i.e.*, all the elements of W are also the elements of V, therefore, we have

$$x + y = y + x\ .\ \forall\ x, y \in W$$

$$(x + y) + z = x + (y + z)\ \ \forall\ x, y, z \in W$$

$$\alpha\,(x + y) = \alpha\, x + \alpha\, y\ \ \forall\ \alpha \in F,\ x, y \in W$$

$$(\alpha + \beta)\, x = \alpha\, x + \beta\, x\ \ \forall\ \ \alpha, \beta \in F,\ x \in W$$

$$1 . x = x\ \ \ \forall\ \ x \in W,\ 1 \in F.$$

Thus W satisfies all the axioms for a vector space

$\Rightarrow$ W is a vector space over F

Hence W is a subspace of V (F).

(*b*) Let R be the field of reals and V be the set of vectors in a plane. Show that V (R) is vector space with vector addition as internal binary composition and scalar multiplication of the elements of R with those of V as external binary composition.

(6½, 7)

Sol. Given $\mathbf{R}^2 = R \times R = \{(x, y) \mid x, y \in R\}$

$$\mathbf{R}^2 = \{(x, y) \mid x, y \in R\}$$

(The elements of $\mathbf{R}^2$ are ordered pairs as $\mathbf{R}^2$ is a set of vectors in a plane)

Here, we define addition of vectors in $\mathbf{R}^2$ as $(x, y) + (t, z) = (x + t, y + z)$. for $x, y, t, z \in R$ and the scalar multiplication of $\alpha \in R$ and $(x, y) \in \mathbf{R}^2$ as $\alpha\,(x, y) = (\alpha\, x, \alpha\, y)$.

I. Properties under addition

A-1 Closure. Let $(x_1, y_1), (x_2, y_2) \in \mathbf{R}^2$

$\Rightarrow x_1, y_1, x_2, y_2 \in R$

$\Rightarrow x_1 + x_2, y_1 + y_2 \in R$

$\therefore (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

$\in \mathbf{R}^2.$

$\Rightarrow \mathbf{R}^2$ is closed under addition.

A-2 Associative. Let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathbf{R}^2$

Now $[(x_1, y_1) + (x_2, y_2)] + (x_3, y_3)$

$= [(x_1 + x_2, y_1 + y_2)] + (x_3\ y_3)$

$= ((x_1 + x_2) + x_3, (y_1 + y_2) + y_3)$

$= (x_1 + (x_2 + x_3), y_1 + (y_2 + y_3))$ [$\because$ Associative Property hold in Reals]

$= (x_1, y_1) + (x_2 + x_3, y_2 + y_3)$

$= (x_1, y_1) + [(x_2, y_2) + (x_3, y_3)]$

$\Rightarrow$ addition is associative in $\mathbf{R}^2$.

A-3 Existence of addative identity.

For all $(x_1, y_1) \in \mathbf{R}^2$, there exists $(0, 0) \in \mathbf{R}^2$

such that $(x_1, y_1) + (0, 0) = (x_1 + 0,\ y_1 + 0)$

$= (x_1, y_1)$

and $(0, 0) + (x_1, y_1) = (0 + x_1, 0 + y_1)$

$= (x_1, y_1)$

$\Rightarrow$ $(0, 0)$ is addative identity in $\mathbf{R}^2$.

A-4 Existence of addative inverse.

Let (x, y) be any element of $\mathbf{R}^2$

$\Rightarrow (-x, -y) \in \mathbf{R}^2$ [Since $x, y \in$ Reals $\Rightarrow -x, -y \in$ Reals]

Now $(x, y) + (-x, -y) = (x + (-x),\ y + (-y))$

$= (0, 0)$

and $(-x, -y) + (x, y) = (-x + x, -y + y)$

$= (0, 0)$

$\therefore (x, y) + (-x, -y) = (0, 0) = (-x, -y) + (x, y)$

$\Rightarrow$ $(-x, -y)$ is the addative inverse of (x, y) for each $(x, y) \in \mathbf{R}^2$.

A-5 Commutative. Let $(x_1, y_1), (x_2, y_2) \in \mathbf{R}^2$

Now $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2,\ y_1 + y_2)$

$= (x_2 + x_1,\ y_2 + y_1)$

[$\because$ addition is commutative in reals]

$= (x_2, y_2) + (x_1,\ y_1)$

$\Rightarrow$ addition is commutative in $\mathbf{R}^2$.

II. Properties under scalar multiplication.

M-1 Let $\alpha \in R$, $(x, y) \in \mathbf{R}^2$; $x, y \in R$

Then $\alpha\,(x, y) = (\alpha\, x, \alpha\, y)$

$\in \mathbf{R}^2$

[$\because$ $\alpha \in R$ and $x, y \in R$ $\Rightarrow \alpha\, x, \alpha\, y \in R$]

M-2 Let $\alpha \in R$ and $(x_1, y_1), (x_2, y_2) \in \mathbf{R}^2$

Now $\alpha\,[(x_1, y_1) + (x_2, y_2)]$

$= \alpha\,[(x_1 + x_2,\ y_1 + y_2)]$

$= (\alpha\,(x_1 + x_2),\ \alpha\,(y_1 + y_2))$

$= (\alpha\, x_1 + \alpha\, x_2,\ \alpha\, y_1 + \alpha\, y_2)$

$= (\alpha\, x_1, \alpha\, y_1) + (\alpha\, x_2, \alpha\, y_2)$

$= \alpha\ (x_1, y_1) + \alpha\,(x_2, y_2).$

M-3 Let $\alpha, \beta \in R$ and $(x_1, y_1) \in \mathbf{R}^2$

Now $(\alpha + \beta)\,(x_1, y_1) = ((\alpha + \beta)\, x_1, (\alpha + \beta)\, y_1)$

$= (\alpha\, x_1 + \beta\, x_1,\ \alpha\, y_1 + \beta\, y_1)$

$= (\alpha\, x_1, \alpha\, y_1) + (\beta\, x_1, \beta\, y_1)$

$= \alpha\,(x_1, y_1) + \beta\,(x_1, y_1).$

M-4 Let $\alpha, \beta \in R$ and $(x_1, y_1) \in \mathbf{R}^2$

Now $(\alpha\,\beta)\,(x, y) = ((\alpha\,\beta)\, x, (\alpha\,\beta)\, y)$

$= (\alpha\,(\beta\, x),\ \alpha\,(\beta\, y)) = \alpha\,(\beta\, x, \beta\, y)$

$= \alpha\,(\beta\,(x, y)).$

M-5 Let $1 \in R$ and $(x_1,\ y_1) \in \mathbf{R}^2$

Now $1.\,(x_1, y_1) = (1\,.\,x_1, 1\,.\,y_1)$

$= (x_1,\ y_1)$

Hence $\mathbf{R}^2$ is a vector space over R.

7. (*a*) For what value of k, will the vector $V = (1, k, -4)$ be a linear combination of $V_1 = (1, -3, 2)$ and $V_2 = (2, -1, 1)$.

Sol. If $V = (1, k, -4)$ is a linear combination of $V_1 = (1, -3, 2)$ and $V_2 = (2, -1, 1)$

Then $V = \alpha_1 V_1 + \alpha_2 V_2$ for some scalars α_1 and α_2

$$\Rightarrow \quad (1, k, -4) = \alpha_1 (1, -3, 2) + \alpha_2 (2, -1, 1) = (\alpha_1, -3\alpha_1, 2\alpha_1) + (2\alpha_2, -\alpha_2, \alpha_2)$$

$$= (\alpha_1 + 2\alpha_2, -3\alpha_1 - \alpha_2, 2\alpha_1 + \alpha_2)$$

By equality of vectors, we get,

$$\alpha_1 + 2\alpha_2 = 1 \quad \ldots(1)$$

$$-3\alpha_1 - \alpha_2 = k \quad \ldots(2)$$

$$2\alpha_1 + \alpha_2 = -4 \quad \ldots(3)$$

Multiplying (3) by 2 and subtracting it from (1), we get,

$$\alpha_1 + 2\alpha_2 - 4\alpha_1 - 2\alpha_2 = 1 + 8$$

$$\Rightarrow \quad -3\alpha_1 = 9 \Rightarrow \alpha_1 = -3$$

Put this value in (1), we get,

$$-3 + 2\alpha_2 = 1 \Rightarrow 2\alpha_2 = 4 \Rightarrow \alpha_2 = 2$$

Putting the values of α_1 and α_2 in (2), we get,

$$-3(-3)(-2) = k \quad \Rightarrow \quad k = 7.$$

(*b*) Prove that $B = \{(1, 1, 0, 0)\ (0, 1, 1, 0)\ (0, 0, 1, 1)\ (1, 0, 0, 0)\}$ is a basis of R^4 and determine the co-ordinate of (2, 3, 4 –1) relative to the ordered basis B.

(6½, 7)

Sol. As dim $R^4 = 4$, thus to show given four vectors form a bases of R^4, it is sufficient to check these vectors are L.I over R

Let $a(1, 1, 0, 0) + b(0, 1, 1, 0) + c(0, 0, 1, 1) + d(1, 0, 0, 0) = (0, 0, 0, 0)$

$\Rightarrow \quad (a + d, a + b, b + c, c) = (0, 0, 0, 0)$

$\therefore \quad a + d = 0,\ a + b = 0,\ b + c = 0,\ c = 0$

$\Rightarrow \quad a = b = c = d = 0$

$\therefore$ given vectors are L.I over R

Hence the given vectors form a basis of R^4 (R).

IInd Part : Let $x, y, z, w \in R$ such that

$$(2, 3, 4, -1) = x(1, 1, 0, 0) + y(0, 1, 1, 0) + z(0, 0, 1, 1) + w(1, 0, 0, 0)$$

$$= (x + w,\ x + y,\ y + z,\ z)$$

$\therefore \quad x + w = 2,\ x + y = 3,\ y + z = 4,\ z = -1$

$\Rightarrow \quad z = -1,\ y = 5,\ x = -2,\ w = 4$

$\therefore \quad (2, 3, 4, -1) = -2(1, 1, 0, 0) + 5(0, 1, 1, 0) + (-1)(0, 0, 1, 1) + 4(1, 0, 0, 0)$

$\Rightarrow$ required coordinate vector is $(-2, 5, -1, 4)$.

Unit-IV

8. (*a*) Determine the eigen values and eigen vectors of the matirx

$$\begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix}.$$

Sol. Let
$$A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 3-\lambda & 1 & 1 \\ 2 & 4-\lambda & 2 \\ 1 & 1 & 3-\lambda \end{bmatrix}$$

∴ the characteristic equation of A is $|A - \lambda I| = 0$

or
$$\begin{vmatrix} 3-\lambda & 1 & 1 \\ 2 & 4-\lambda & 2 \\ 1 & 1 & 3-\lambda \end{vmatrix} = 0$$

or
$$\begin{vmatrix} 6-\lambda & 6-\lambda & 6-\lambda \\ 2 & 4-\lambda & 2 \\ 1 & 1 & 3-\lambda \end{vmatrix} = 0, \text{ by } R_1 \to R_1 + R_2 + R_3$$

or
$$(6-\lambda)\begin{vmatrix} 1 & 1 & 1 \\ 2 & 4-\lambda & 2 \\ 1 & 1 & 3-\lambda \end{vmatrix} = 0 \quad \text{or} \quad (6-\lambda)\begin{vmatrix} 1 & 0 & 0 \\ 2 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0$$

or $(6-\lambda)\,[(1)(2-\lambda)(2-\lambda)] = 0$

or
$$(6-\lambda)(2-\lambda)^2 = 0$$

∴ $\lambda = 2, 2, 6$

which are the eigen values of A.

The eigen vector $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq O$ corresponding to the eigen value $\lambda = 6$ is given by

$$AX = \lambda X \quad \text{or} \quad (A - 6\,I)\,X = O$$

or
$$\begin{bmatrix} -3 & 1 & 1 \\ 2 & -2 & 2 \\ 1 & 1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 1 & -3 \\ 2 & -2 & 2 \\ -3 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{or} \begin{bmatrix} 1 & 1 & -3 \\ 0 & -4 & 8 \\ 0 & 4 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ by } R_2 \to R_2 - 2R_1,\ R_3 \to R_3 + 3R_1$$

$$\text{or} \begin{bmatrix} 1 & 1 & -3 \\ 0 & -4 & 8 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ by } R_3 \to R_3 + R_2$$

Now the coefficient matrix of these equations is of rank 2. Therefore these equations have only $3 - 2 = 1$ L.I. solution. Thus there is only one L.I. eigen vector corresponding to the value 6. These equations can be written as

$$x + y - 3z = 0$$

$$-4y + 8z = 0 \quad \Rightarrow y = 2z$$

$$\therefore\ x + 2z - 3z = 0 \quad \Rightarrow x = z$$

Take $z = 1,\quad \therefore\ x = 1, y = 2$

$$X = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \text{ is an eigen vector of A.}$$

The eigen vector $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq O$ corresponding to the eigen value $\lambda = 2$ is given by

$$AX = 2X \quad \text{or} \quad (A - 2I)X = O$$

$$\text{or} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{or} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ by } R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$$

The coefficient matrix of these equations is of rank 1. Therefore these equations have $3 - 1 = 2$ L.I. solutions. These equations can be written as

$$x + y + z = 0 \quad \text{or} \quad x = -y - z$$

Take $y = 1, z = 0 \quad ; \quad y = 0, z = 1.$

Therefore we find two L.I. eigen vectors of A as $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

(*b*) Let $T : R^3 \to R^2$ be defined by $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y-z \\ 2x-y+z \end{bmatrix}$, then find the matrix associated with this linear transformation. (6½, 7)

Sol. If $T : R^3 \to R^2$ be defined by

$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y-z \\ 2x-y+z \end{bmatrix}$. We are to find the matrix associated with this transformation

Consider the usual basis for R^3

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \; T\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ and } T\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Then the matrix A is given by $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}$.

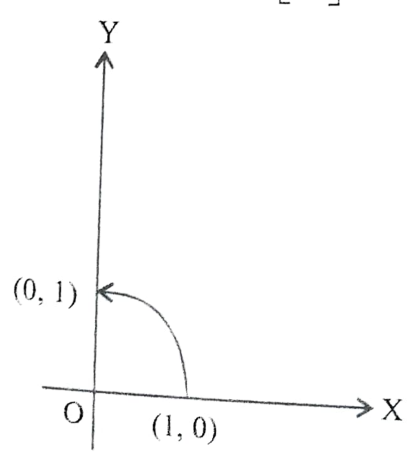
9. (*a*) Find a matrix representation for counterclockwise rotation of the plane about origin through 90°.

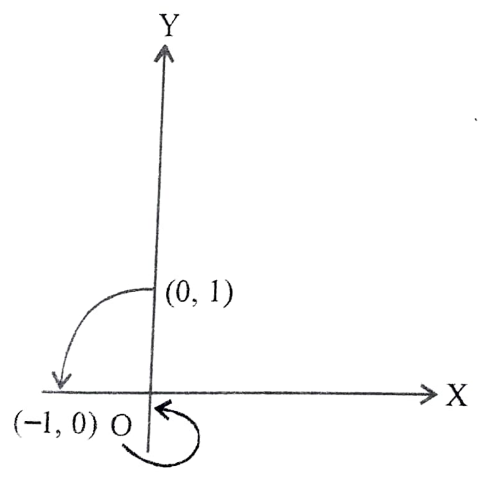
Sol. Let $T:R^2 \to R^2$ be a linear transformation denoting rotation through 90°.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix associated with this transformation

s.t. $$T(X) = AX \;\forall\; X \in R^2$$

As we know that the rotation through 90° will transform the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ on the *x*-axis to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ on the *y*-axis to $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$





Also $T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$ and $T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-1\\0\end{bmatrix}$

But we know that T (X) = AX so

$$\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} \text{ and } \begin{bmatrix}-1\\0\end{bmatrix} = \begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}$$

On solving these equations, we have

$$\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}a\\c\end{bmatrix} \text{ and } \begin{bmatrix}-1\\0\end{bmatrix} = \begin{bmatrix}b\\d\end{bmatrix}$$

⇒ $a = 0,\ b = -1, c = 1, d = 0$

So matrix $A = \begin{bmatrix}a & b\\c & d\end{bmatrix} = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$ represents rotation of plane through 90°.

(*b*) Let $T : R^2 \to R^2$ be the projection onto the *x*-axis. Find the eigen values and the corresponding eigenvectors for T. Interpret them geometrically. (6½, 7)

Sol. As we know that projection onto *x*-axis is given by

$T\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x\\0\end{bmatrix}$ and the matrix associated with this transformation is

$$A = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$$

$$I = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \qquad \Rightarrow \lambda I = \begin{bmatrix}\lambda & 0\\0 & \lambda\end{bmatrix}$$

The characteristics equation is $|A - \lambda I| = 0$

or $\begin{vmatrix}1-\lambda & 0\\0 & -\lambda\end{vmatrix} = 0 \Rightarrow (-\lambda)(1-\lambda) = 0 \Rightarrow \lambda = 0, 1$

The egien vector $X = \begin{bmatrix}x\\y\end{bmatrix} \neq O$ corresponding to the eigen value $\lambda = 0$ is given by

$AX = O$

$\therefore \quad \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

∴ $x = 0$, but y is independent so it can be any value, so

$X = \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}0\\k\end{bmatrix}$ is the associated eigenvector.

The egien vector $X = \begin{bmatrix} x \\ y \end{bmatrix} \neq O$ corresponding to the eigen value $\lambda = 1$ is given by

$$AX = X \text{ or } (A - I)X = O$$

$$\therefore \quad \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore \quad y = 0$, but x can take any values so

$x = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k \\ 0 \end{bmatrix}$ be its eigenvector, k is any real no.

If X is a vector along x-axis, then it is of the form $\begin{bmatrix} k \\ 0 \end{bmatrix}$ for some real k,

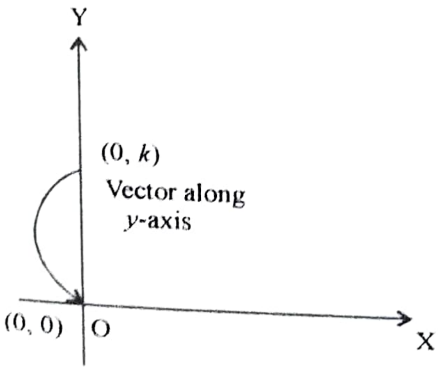
so $T\begin{bmatrix} k \\ 0 \end{bmatrix} = \begin{bmatrix} k \\ 0 \end{bmatrix} = 1\begin{bmatrix} k \\ 0 \end{bmatrix}$

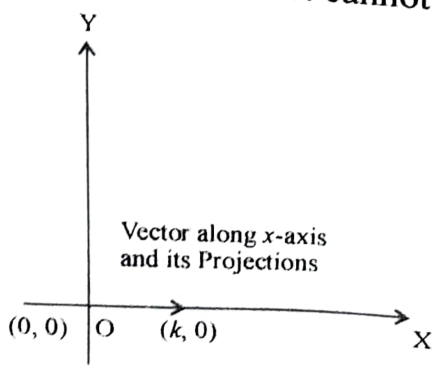
geometrically, the projection of any vector along x-axis is a vector itself. Thus 1 is an eigenvalue of T and the corresponding eigenvectors are the non zero vector along x-axis.

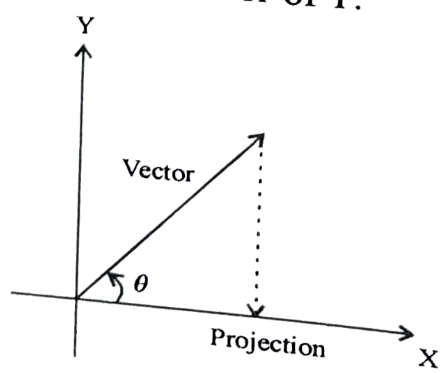
If X is a vector along y-axis it is of the form $\begin{bmatrix} 0 \\ k \end{bmatrix}$ and $T\begin{bmatrix} 0 \\ k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \cdot \begin{bmatrix} 0 \\ k \end{bmatrix}$

i.e. T projects the vectors along y-axis onto the zero vector. So 0 is an eigenvalue and non-zero vectors along y-axis are the corresponding eigenvectors.

If X is any other vector neither parallel to x-axis nor parallel to y-axis, then the projection T(X) of X on x-axis is not a scalar multiple of a given vector. Thus, such a projection is neither zero nor parallel to x-axis. So it cannot be an eigen vector of T.







Zero vector is its projection.