HIMACHAL PRADESH UNIVERSITY-2021

SOLVED PAPER

B.A./B.SC.-III MATHEMATICS
NUMERICAL METHODS

Time : 3 Hours] **[Maximum Marks : 70**

Note : Section A is compulsory. Attempt four questions from Section-B, selecting one each from Units I, II, III and IV. Use of non-scientific/non-programmable calculator is allowed.

SECTION – A

Compulsory Question

1. (*i*) Give any two advantages of Bisection Method.

Sol. (*i*) The bisection method is simple to use.

(*ii*) Convergence is assured in the bisection method for any $f(x)$ which is continuous in the interval containing the root.

(*iii*) This method is suitable for implementation on a computer.

(*ii*) Define Transcendental Equation.

Sol. If the equation $f(x) = 0$ involves transcendental functions such as e^x, $\log x$, $\sin x$ etc. then it is called a *transcendental equation.*

e.g. $x + \cos x = 0,\ xe^x - 1 = 0$ etc.

(*iii*) What is LU decomposition Method ?

Sol. Suppose that we have to solve a linear system

$$AX = B \quad ...(1)$$

We can express the matrix A as a product of a lower triangular matrix L and an upper triangular matrix U. So we can write matrix A as

$$A = LU \quad ...(2)$$

where $L = \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix}$ and $U = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{bmatrix}$

The form (2) is called an LU decomposition of A.

(*iv*) How many finite differences are there ? Name them.

Sol. Three, (1) Forward Differences (2) backward Differences (3) Central Differences.

(v) Write Newton's forward difference interpolation formula.

Suppose that the function f is tabulated at $(n + 1)$ equidistant points $x_0, x_1, x_2, ..., x_n$ with spacing h and the corresponding values of the function f are $y_0, y_1, y_2, ..., y_n$ respectively.

In order to derive Newton's forward difference formula, we approximate the given function by a polynomial $\phi_n(x)$ of degree n such that $f(x)$ and $\phi_n(x)$ agree at the tabulated points.

So $\quad f(x) \approx \phi_n(x) \quad$ ...(1)

and $\quad \phi_n(x_i) = y_i$ for $i = 0, 1, 2, 3, ..., n \quad$ ...(2)

$$\text{Then } \phi_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots + \frac{p(p-1)(p-2)\ldots(p-\overline{n-1})}{n!}\Delta^n y_0 \quad \ldots(3)$$

Using (1), equation (5) can also be written as

$$f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots + \frac{p(p-1)(p-2)\ldots(p-\overline{n-1})}{n!}\Delta^n y_0 \quad \ldots(4)$$

Formula (3) (or (4) is called *Newton's forward difference formula* or *Newton-Gregory forward difference formula.*

(vi) Which rule gives the exact value of the integral if $f(x)$ is a quadratic equation ?

Sol. Simpson's $\frac{1}{3}$ Rule

(vii) Define Numerical Integration.

Sol. *Numerical integration* is the process of calculating the value of a definite integral from a set of tabulated values of the integrand.

(viii) Define Balzano Method.

Sol. Balzano method for finding the root of the equation $f(x) = 0$ is based on the repeated application of intermediate value theorem which states that if f is a continuous function on the interval $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs then there exists atleast one real root of $f(x) = 0$ in the interval (a, b).

Without loss of generality, suppose that a continuous function f is negative at a and positive at b, so there exists atleast one real root between a and b. (We may also take f as positive at a and negative at b).

Let the approximate value of root be $x_1 = \frac{a+b}{2}$ *i.e.* the point of bisection of the interval (a, b). Now, if we evaluate $f(x_1)$, there are three possibilities :

(i) $f(x_1) = 0$, in which case x_1 is the root.

(*ii*) $f(x_1) < 0$, in which case the root lies in the interval (x_1, b)

(*iii*) $f(x_1) > 0$, in which case the root lies in the interval (a, x_1)

Presuming there is just one root, if case (*i*) occurs, the process is terminated. If either case (*ii*) or case (*iii*) occurs, the process of bisection of the interval containing the root can be repeated until the root is obtained to the desired accuracy. The bisection method is shown graphically in fig. 1 in which the successive points of bisection are denoted by x_1, x_2 and x_3.

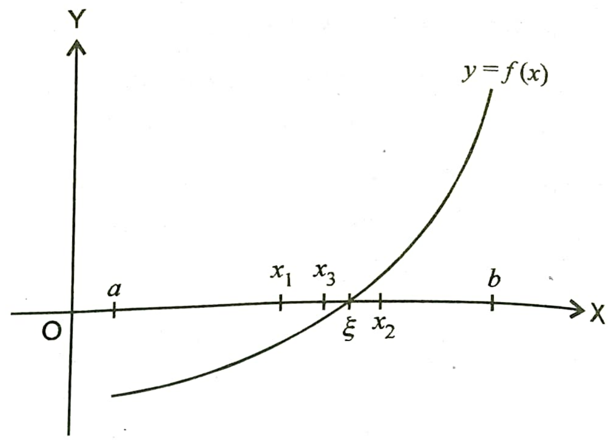

Fig. 1

(2 × 8 = 16)

SECTION – B

Unit-I

2. (*a*) Find a root of the equation $x^3 - 5x + 3 = 0$ between 1.75 and 2 correct to three decimal places using the bisection method.

Sol. The given equation is $x^3 - 5x + 3 = 0$

Let $f(x) = x^3 - 5x + 3.$

Now $f(1.75) = (1.75)^3 - 5(1.75) + 3 = -0.3906 < 0$

and $f(2) = 2^3 - 5(2) + 3 = 1 > 0$

So a real root of given equation lies in the interval (1.75, 2)

Iteration 1. Taking $a = 1.75$ and $b = 2$.

The first approximation to the root is given by $x_1 = \frac{a+b}{2} = \frac{1.75+2}{2} = 1.875$

Now $f(1.875) = (1.875)^3 - 5(1.875) + 3 = 0.2168 > 0$

and $f(1.75) = -0.3906 < 0$

So a real root of given equation lies in the interval (1.75, 1.875)

Iteration 2. Taking $a = 1.75$ and $b = 1.875$

The second approximation to the root is given by $x_2 = \frac{a+b}{2} = \frac{1.75+1.875}{2} = 1.8125$

Now $f(1.8125) = (1.8125)^3 - 5(1.8125) + 3 = -0.1081 < 0$

and $f(1.875) = 0.2168 > 0$

So a real root of given equation lies in the interval (1.8125, 1.875)

Iteration 3. Taking $a = 1.8125$ and $b = 1.875$

The third approximation to the root is given by $x_3 = \frac{a+b}{2} = \frac{1.8125+1.875}{2} = 1.8438$

Now $f(1.8438) = (1.8438)^3 - 5(1.8438) + 3 = 0.0492 > 0$

and $f(1.8125) = -0.1081 < 0$

So a real root of given equation lies in the interval (1.8125, 1.8438)

Similarly, by performing subsequent iterations, the successive approximations to the root are given by

$$x_4 = 1.8281,\ x_5 = 1.8360,\ x_6 = 1.8320,\ x_7 = 1.8340,\ x_8 = 1.8350,\ x_9 = 1.8345,\ x_{10} = 1.8342.$$

From 9th and 10th iteration, we see that there is no change in the successive approximations to the root upto first three decimal places.

So a real root of given equation is given by $x = 1.834$ (correct to first three decimal places).

(*b*) Find a real root of equation $x^3 - 3x - 5 = 0$ by Newton Raphson method. **(6½, 7)**

Sol. The given equation is $x^3 - 3x - 5 = 0$

Let $f(x) = x^3 - 3x - 5$

$\therefore$ $f'(x) = 3x^2 - 3$

Now $f(2) = (2)^3 - 3(2) - 5 = -3 < 0$

and $f(3) = (3)^3 - 3(3) - 5 = 13 > 0$

So a real root of given equation lies between 2 and 3

Also $f(2)$ is nearer to zero than $f(3)$ so take initial approximation x_0 to the root as 2

Iteration 1. The first approximation to the root is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{x_0^3 - 3x_0 - 5}{3x_0^2 - 3} = 2 - \frac{[2^3 - 3(2) - 5]}{3(2)^2 - 3} = 2.3333$$

Iteration 2. The second approximation to the root is given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 - \frac{x_1^3 - 3x_1 - 5}{3x_1^2 - 3} = 2.3333 - \frac{[(2.3333)^2 - 3(2.3333) - 5]}{3(2.3333)^2 - 3}$$

$$= 2.2806$$

Iteration 3. The third approximation to the root is given by

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = x_2 - \frac{x_2^3 - 3x_2 - 5}{3x_2^2 - 3} = 2.2806 - \frac{[(2.2806)^3 - 3(2.2806) - 5]}{3(2.2806)^2 - 3}$$

$$= 2.2790$$

Iteration 4. The fourth approximation to the root is given by

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = x_3 - \frac{x_3^3 - 3x_3 - 5}{3x_3^2 - 3} = 2.2790 - \frac{[(2.2790)^3 - 3(2.2790) - 5]}{3(2.2790)^2 - 3}$$

$$= 2.2790$$

From 3rd and 4th iteration, we observe that the successive approximations to the root are same. So we stop the iterative procedure. So a real root of given equation is given by $x = 2.2790$

3. (*a*) Find a root of equation $x^4 - x - 10 = 0$ using secant method.

Sol. The given equation is $x^4 - x - 10 = 0$

Let $f(x) = x^4 - x - 10$

Now $f(0) = -10 < 0,\ f(0.5) = (0.5)^4 - 0.5 - 10 = -10.4375 < 0$

$f(1) = 1 - 1 - 10 = -10 < 0$

$f(1.5) = (1.5)^4 - 1.5 - 10 = -6.4375 < 0$

and $f(2) = 2^4 - 2 - 10 = 4 > 0$

So a real root of given equation lies between 1.5 and 2.

Iteration 1. Taking $a = 1.5$ and $b = 2$ so that $f(a) = -6.4375$ and $f(b) = 4$

The first approximation to the root is given by

$$x_1 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)} = \frac{1.5(4) - 2(-6.4375)}{4 - (-6.4375)} = 1.8084$$

Iteration 2. Taking $a = 2$ and $b = 1.8084$ so that $f(a) = 4$

and $f(b) = (1.8084)^4 - 1.8084 - 10 = -1.1135$

The second approximation to the root is given by

$$x_2 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)} = \frac{2(-1.1135) - 1.8084(4)}{-1.1135 - 4} = 1.8501$$

Iteration 3. Taking $a = 1.8084$ and $b = 1.8501$ so that $f(a) = -1.1135$

and $f(b) = (1.8501)^4 - 1.8501 - 10 = -0.1341$

The third approximation to the root is given by

$$x_3 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)} = \frac{1.8084(-0.1341) - 1.8501(-1.1135)}{-0.1341 - (-1.1135)} = 1.8558$$

Iteration 4. Taking $a = 1.8501$ and $b = 1.8558$ so that $f(a) = -0.1341$

and $f(b) = (1.8558)^4 - 1.8558 - 10 = 0.0053$

The fourth approximation to the root is given by

$$x_4 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)} = \frac{1.8501(0.0053) - 1.8558(-0.1341)}{0.0053 - (-0.1341)} = 1.8556$$

We observe that in 3rd and 4th iteration, the successive approximations to the root are approximately same. So we stop the iteration procedure. So a real of given equation is given by $x = 1.8556$.

(b) Using L U Decomposition, solve the equations

$$2x + y + 2z = 2$$

$$x + y + 3z = 4$$

$$x + y + z = 0$$

(6½, 7)

Sol. Given equations can be written as A X = B ...(1)

where $A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 5 & 3 \\ 1 & 1 & -1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$

Let $A = LU$ ...(2)

where $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}, U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$

$$\therefore \begin{bmatrix} 2 & 1 & 2 \\ 1 & 5 & 3 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 1 & 2 \\ 1 & 5 & 3 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11} u_{12} & l_{11} u_{13} \\ l_{21} & l_{21} u_{12} + l_{22} & l_{21} u_{13} + l_{22} u_{23} \\ l_{31} & l_{31} u_{12} + l_{32} & l_{31} u_{13} + l_{32} u_{23} + l_{33} \end{bmatrix}$$

Comparing the corresponding elements on both sides, we get,

First column :

$$l_{11} = 2, \ l_{21} = 1, \ l_{31} = 1$$

First row :

$$l_{11} u_{12} = 1, \ l_{11} u_{13} = 2$$

$$\therefore \ u_{12} = \frac{1}{2}, \ u_{13} = 1$$

Second column :

$$l_{21} u_{12} + l_{22} = 5 \qquad \Rightarrow (1) \times \frac{1}{2} + l_{22} = 5$$

$$\therefore \ l_{22} = 5 - \frac{1}{2} = \frac{9}{2}$$

$$l_{31}\, u_{12} + l_{32} = 1 \qquad \Rightarrow (1) \times \frac{1}{2} + l_{32} = 1$$

$$\therefore \quad l_{32} = \frac{1}{2}$$

Second row :

$$l_{21}\, u_{13} + l_{22}\, u_{23} = 3$$

$$1 \times 1 + \frac{9}{2} \times u_{23} = 3$$

$$\therefore \quad u_{23} = \frac{2}{9}\,(3 - 1) = \frac{4}{9}$$

Third column :

$$l_{31}\, u_{13} + l_{32}\, u_{23} + l_{33} = -1$$

$$1 \times 1 + \frac{1}{2} \times \frac{4}{9} + l_{33} = -1$$

$$\therefore \quad l_{33} = -1 - 1 - \frac{4}{18} = -\frac{20}{9}$$

$$\therefore \quad L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & \frac{9}{2} & 0 \\ 1 & \frac{1}{2} & \frac{-20}{9} \end{bmatrix}, \quad U = \begin{bmatrix} 1 & \frac{1}{2} & 1 \\ 0 & 1 & \frac{4}{9} \\ 0 & 0 & 1 \end{bmatrix}$$

From (1) and (2), we have

$$(L\,U)\,X = B \qquad \text{or} \qquad L\,(U\,X) = B$$

Putting $U\,X = Y$ where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, we have

$$L\,Y = B \qquad \text{or} \qquad \begin{bmatrix} 2 & 0 & 0 \\ 1 & \frac{9}{2} & 0 \\ 1 & \frac{1}{2} & \frac{-20}{9} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad 2y_1 = 2 \qquad \therefore \quad y_1 = 1$

$y_1 + \frac{9}{2}y_2 = 4 \qquad \therefore \quad y_2 = \frac{2}{9}(4 - y_1) = \frac{2}{9}(4-1) = \frac{2}{3}$

$y_1 + \frac{1}{2}y_2 - y_3 = 0$

$$y_3 = \frac{9}{20}\left(y_1 + \frac{1}{2}y_2\right) = \frac{9}{20}\left(1 + \frac{1}{2} \times \frac{2}{3}\right) = \frac{9}{20} \times \frac{4}{3} = \frac{3}{5}$$

From (3), $\quad UX = Y$

$$\begin{bmatrix} 1 & \frac{1}{2} & 1 \\ 0 & 1 & \frac{4}{9} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{2}{3} \\ \frac{3}{5} \end{bmatrix}$$

$\Rightarrow \quad x + \frac{1}{2}y + z = 1 \qquad \ldots(4)$

$y + \frac{4}{9}z = \frac{2}{3} \qquad \ldots(5)$

$z = \frac{3}{5} \qquad \ldots(6)$

From (5), $\quad y + \frac{4}{9} \times \frac{3}{5} = \frac{2}{3}$

or $\quad y = \frac{2}{3} - \frac{4}{15} = \frac{10-4}{15} = \frac{2}{5}$

From (4), $\quad x + \frac{1}{2} \times \frac{2}{5} + \frac{3}{5} = 1$

or $\quad x = 1 - \frac{1}{5} - \frac{3}{5} = \frac{5-1-3}{5} = \frac{1}{5}$

$\therefore$ The solution is $x = \frac{1}{5}$, $y = \frac{2}{5}$ and $z = \frac{3}{5}$

Unit-II

4. (*a*) Solve the following system of equations correct to four significant digits, by Jacobi method.

$$20x + y - 2z = 17\ ;\ 3x + 20y - z = -18\ ;\ 2x - 3y + 20x = 25$$

Sol. The given system of equations is a diagonal system so the convergence of Jacobi method is assured.

Rewriting the given system of equations as follows :

$$x = \frac{1}{20}(17 - y + 2z)$$

$$y = \frac{1}{20}(-18 - 3x + z)$$

$$z = \frac{1}{20}(25 - 2x + 3y)$$

Let us take the initial approximation to each unknown as zero *i.e.* $x^{(0)} = y^{(0)} = z^{(0)} = 0$

Iteration 1. Putting the initial values in right side of (1), we get

$$x^{(1)} = \frac{1}{20}\left(17 - y^{(0)} + 2z^{(0)}\right) = \frac{1}{20}(17 - 0 + 2(0)) = 0.85$$

$$y^{(1)} = \frac{1}{20}\left(-18 - 3x^{(0)} + z^{(0)}\right) = \frac{1}{20}(-18 + 3(0) + 0) = -0.9$$

$$z^{(1)} = \frac{1}{20}\left(25 - 2x^{(0)} + 3y^{(0)}\right) = \frac{1}{20}(25 - 2(0) + 3(0)) = 1.25$$

Iteration 2. $x^{(2)} = \frac{1}{20}\left(17 - y^{(1)} + 2z^{(1)}\right) = \frac{1}{20}(17 + 0.9 + 2(1.25)) = 1.02$

$$y^{(2)} = \frac{1}{20}\left(-18 - 3x^{(1)} + z^{(1)}\right) = \frac{1}{20}(-18 - 3(0.85) + 1.25) = -0.965$$

$$z^{(2)} = \frac{1}{20}\left(25 - 2x^{(1)} + 3y^{(1)}\right) = \frac{1}{20}(25 - 2(0.85) + 3(-0.9)) = 1.03$$

Iteration 3. $x^{(3)} = \frac{1}{20}\left(17 - y^{(2)} + 2z^{(2)}\right) = \frac{1}{20}(17 + 0.965 + 2(1.03)) = 1.0012$

$$y^{(3)} = \frac{1}{20}\left(-18 - 3x^{(2)} + z^{(2)}\right) = \frac{1}{20}(-18 - 3(1.02) + 1.03) = -1.0015$$

$$z^{(3)} = \frac{1}{20}\left(25 - 2x^{(2)} + 3y^{(2)}\right) = \frac{1}{20}(25 - 2(1.02) + 3(-0.965)) = 1.0032$$

Iteration 4. $x^{(4)} = \frac{1}{20}\left(17 - y^{(3)} + 2z^{(3)}\right) = \frac{1}{20}(17 + 1.0015 + 2\,(1.0032)) = 1.0004$

$$y^{(4)} = \frac{1}{20}\left(-18 - 3x^{(3)} + z^{(3)}\right) = \frac{1}{20}(-18 - 3\,(1.0012) + 1.0032) = -\,1.0000$$

$$z^{(4)} = \frac{1}{20}\left(25 - 2x^{(3)} + 3y^{(3)}\right) = \frac{1}{20}(25 - 2\,(1.0012) + 3\,(-\,1.0015)) = 0.9996$$

Iteration 5. $x^{(5)} = \frac{1}{20}\left(17 - y^{(4)} + 2z^{(4)}\right) = \frac{1}{20}(17 + 1.0000 + 2\,(0.9996)) = 1.0000$

$$y^{(5)} = \frac{1}{20}\left(-18 - 3x^{(4)} + z^{(4)}\right) = \frac{1}{20}(-18 - 3\,(1.0004) + 0.9996) = -\,1.0000$$

$$z^{(5)} = \frac{1}{20}\left(25 - 2x^{(4)} + 3y^{(4)}\right) = \frac{1}{20}(25 - 2\,(1.0004) + 3\,(-\,1.0000)) = 1.0000$$

Iteration 6. $x^{(6)} = \frac{1}{20}\left(17 - y^{(5)} + 2z^{(5)}\right) = \frac{1}{20}(17 + 1.0000 + 2\,(1.0000)) = 1.0000$

$$y^{(6)} = \frac{1}{20}\left(-18 - 3x^{(5)} + z^{(5)}\right) = \frac{1}{20}(-18 - 3\,(1.0000) + 1.0000) = -\,1.0000$$

$$z^{(6)} = \frac{1}{20}\left(25 - 2x^{(5)} + 3y^{(5)}\right) = \frac{1}{20}(25 - 2\,(1.0000) + 3\,(-\,1.0000)) = 1.0000$$

We observe that in 5th and 6th iteration, there is no change in first four significant figures in the approximations to the unknowns so we stop the iterative procedure.

Hence the solutions of given system of equations, correct to four significant digits, is given by

$$x = 1.000,\ y = -\,1.000,\ z = 1.000.$$

(*b*) Use four iterations to solve the following system of equations starting with initial solution as $\left(\frac{9}{5}, \frac{4}{5}, \frac{-6}{5}\right)$ by Gauss Seidel method :

$$\begin{bmatrix} 5 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \\ -6 \end{bmatrix}$$ **(6½, 7)**

Sol. The given system of equations is a diagonal system. So convergence of Gauss-Seidel method is assured.

Rewriting the given system of equations as follows :

$$x = \frac{1}{5}(9 + y)$$

$$y = \frac{1}{5}(4 + x + z)$$

$$z = \frac{1}{5}(-6 + y)$$

Given $x^{(0)} = \frac{9}{5},\ y^{(0)} = \frac{4}{5},\ z^{(0)} = -\frac{6}{5}$

Iteration 1. $x^{(1)} = \frac{1}{5}\left(9 + y^{(0)}\right) = \frac{1}{5}\left(9 + \frac{4}{5}\right) = 1.96$

$$y^{(1)} = \frac{1}{5}\left(4 + x^{(1)} + z^{(0)}\right) = \frac{1}{5}\left[4 + 1.96 + \left(-\frac{6}{5}\right)\right] = 0.952$$

$$z^{(1)} = \frac{1}{5}\left(-6 + y^{(1)}\right) = \frac{1}{5}(-6 + 0.952) = -1.0096$$

Iteration 2. $x^{(2)} = \frac{1}{5}\left(9 + y^{(1)}\right) = \frac{1}{5}(9 + 0.952) = 1.9904$

$$y^{(2)} = \frac{1}{5}\left(4 + x^{(2)} + z^{(1)}\right) = \frac{1}{5}(4 + 1.9904 - 1.0096) = 0.9962$$

$$z^{(2)} = \frac{1}{5}\left(-6 + y^{(2)}\right) = \frac{1}{5}(-6 + 0.9962) = -1.0008$$

Iteration 3. $x^{(3)} = \frac{1}{5}\left(9 + y^{(2)}\right) = \frac{1}{5}(9 + 0.9962) = 1.9992$

$$y^{(3)} = \frac{1}{5}\left(4 + x^{(3)} + z^{(2)}\right) = \frac{1}{5}(4 + 1.9992 - 1.0008) = 0.9997$$

$$z^{(3)} = \frac{1}{5}\left(-6 + y^{(3)}\right) = \frac{1}{5}(-6 + 0.9997) = -1.0001$$

Iteration 4. $x^{(4)} = \frac{1}{5}\left(9 + y^{(3)}\right) = \frac{1}{5}(9 + 0.9997) = 1.9999$

$$y^{(4)} = \frac{1}{5}\left(4 + x^{(4)} + z^{(3)}\right) = \frac{1}{5}(4 + 1.9999 - 1.0001) = 1.0000$$

$$z^{(4)} = \frac{1}{5}\left(-6 + y^{(4)}\right) = \frac{1}{5}(-6 + 1.0000) = -1.0000$$

Hence, after four iterations, the solution of given system of equations is

$$x = 1.9999,\ y = 1.0000,\ z = -1.0000$$

5. (*a*) From the following table, Interpolate the value of $y(x)$ using Lagrangian polynomial at 2.8 :

x :	2.0	3.0	4.0
$y(x)$:	6.6	9.2	8.6

Sol. Here $x_0 = 2.0$, $x_1 = 3.0$, $x_2 = 4.0$ and $y_0 = 6.6$, $y_1 = 9.2$, $y_2 = 8.6$.

We know that Lagrangian polynomial is given by

$$y(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

$$= \frac{(x-3)(x-4)}{(2-3)(2-4)} \times 6.6 + \frac{(x-2)(x-4)}{(3-2)(3-4)} \times 9.2 + \frac{(x-2)(x-3)}{(4-2)(4-3)} \times 8.6$$

$$= 3.3\,(x^2 - 7x + 12) - 9.2(x^2 - 6x + 8) + 4.3(x^2 - 5x + 6)$$

$$= -\,1.6x^2 + x(-\,23.1 + 55.2 - 21.5) + (39.6 - 73.6 + 25.8)$$

$$= -\,1.6x^2 + 10.6x - 8.2$$

$\therefore$ $y(2.8) = -1.6\,(2.8)^2 + 10.6\,(2.8) - 8.2 = -12.544 + 29.68 - 8.2 = 8.936$

(*b*) Using Newton's divided difference formula, evaluate $f(x)$ and $f(15)$ from the following data :

x :	4	5	7	10	11	13
$f(x)$:	48	100	294	900	1210	2028

(6½, 7)

Sol. The divided difference table is :

x	$y = f(x)$	$\Delta_d y$	$\Delta_d^2 y$	$\Delta_d^3 y$	$\Delta_d^4 y$
4	48				
		52			
5	100		15		
		97		1	
7	294		21		0
		202		1	
10	900		27		0
		310		1	
11	1210		33		
		409			
13	2028				

By Newton's divided difference formula,

$$f(x) = y_0 + (x - x_0)\Delta_d\, y_0 + (x - x_0)(x - x_1)\Delta_d^2\, y_0 + (x - x_0)(x - x_1)(x - x_2)\Delta_d^3\, y_0$$

$$= 48 + (x - 4)\,52 + (x - 4)(x - 5)\,15 + (x - 4)(x - 5)(x - 7)\,1$$

$$= 48 + 52x - 208 + 15(x^2 - 9x + 20) + (x^3 - 16x^2 + 83x - 140)$$

$$= x^3 - x^2$$

So $\qquad f(x) = x^3 - x^2$

Hence $\qquad f(8) = 8^3 - 8^2 = 448$

and $\qquad f(15) = 15^3 - 15^2 = 3150$

Unit-III

6. (*a*) Construct backward difference table for the following data :

x :	0	1	2	3
$f(x)$:	-3	6	8	12

Evaluate $\nabla^3 f(3)$ and $\nabla^2 f(2)$.

Sol. The backward difference table is :

x	$y = f(x)$	∇y	$\nabla^2 y$	$\nabla^3 y$
0	-3			
		9		
1	6		-7	
		2		9
2	8		2	
		4		
3	12			

As we know that the subscript remains constant along each backward diagonal.

$\therefore \qquad \nabla^3 f(3)$ or $\nabla^3 y_3 = 9$

and $\qquad \nabla^2 f(2)$ or $\nabla^2 y_2 = -7$.

(*b*) The population of a town in the decimal census was as given below. Estimate the population for year 1895 :

Year (x) :	1891	1901	1911	1921	1931
Population (y) (in thousands) :	46	66	81	93	101

(6½, 7)

Sol. The forward difference table is :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1891	46				
		20			
1901	66		-5		
		15		2	
1911	81		-3		-3
		12		-1	
1921	93		-4		
		8			
1931	101				

Here $x_0 = 1891$ so that $y_0 = 46,\ \Delta y_0 = 20,\ \Delta^2 y_0 = -5,\ \Delta^3 y_0 = 2,\ \Delta^4 y_0 = -3.$

Also $h = 10$

$$\therefore \quad p = \frac{x - x_0}{h} = \frac{1895 - 1891}{10} = 0.4$$

By Newton's forward difference formula,

$$y(x) = y_0 + p\,\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$$\therefore \quad y(1895) = 46 + 0.4\,(20) + \frac{0.4\,(0.4-1)}{2}(-5) + \frac{0.4\,(0.4-1)\,(0.4-2)}{6}(2)$$

$$+ \frac{(0.4)\,(0.4-1)\,(0.4-2)\,(0.4-3)}{24}(-3)$$

$$= 46 + 8 + 0.6 + 0.128 + 0.1248$$

$= 54.853$ thousands (rounded off to three decimal places)

7. (*a*) Find the cubic polynomial which takes the following values

$y(0) = 1,\ y(1) = 0,\ y(2) = 1$ and $y(3) = 10$.

Hence or otherwise obtain $y(4)$.

Sol. The difference table is :

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
0	1			
		-1		
1	0		2	
		1		6
2	1		8	
		9		
3	10			

Here take $x_n = 3$ so that $y_n = 10$, $\nabla y_n = 9$, $\nabla^2 y_n = 8$, $\nabla^3 y_n = 6$.

Also $h = 1$

$$\therefore \qquad p = \frac{x - x_n}{h} = \frac{x-3}{1} = x - 3$$

By Newton's backward interpolation formula, we get

$$y(x) = y_n + p\,\nabla y_n + \frac{p\,(p+1)}{2!}\nabla^2 y_n + \frac{p\,(p+1)\,(p+2)}{3!}\nabla^3 y_n$$

$$= 10 + (x-3)\,9 + \frac{(x-3)\,(x-2)}{2} \times 8 + \frac{(x-3)\,(x-2)\,(x-1)}{6} \times 6$$

$$= 10 + 9x - 27 + 4\,(x^2 - 5x + 6) + (x^2 - 5x + 6)\,(x - 1)$$

$$= 10 + 9x - 27 + 4x^2 - 20x + 24 + x^3 - 6x^2 + 11x - 6$$

$= x^3 - 2x^2 + 1$, which is the required cubic polynomial.

Hence $\quad y(4) = (4)^3 - 2\,(4)^2 + 1 = 64 - 32 + 1 = 33.$

(*b*) Use Stirling's formula to evaluate $f(1.22)$, given :

x :	1.0	1.1	1.2	1.3	1.4
$f(x)$:	0.841	0.891	0.932	0.963	0.985

Sol. The difference table is :

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
1.0	0.841			
		0.051		
1.1	0.891		– 0.010	
		0.041		0
1.2	0.932		– 0.010	
		0.031		0.001
1.3	0.963		– 0.009	
		0.022		
1.4	0.985			

It is clear from the table that the third order difference contribution to Stirling's formula is negligible. So we neglect the terms containing third and higher order differences.

Take $x_0 = 1.2$ so that $y_0 = 0.932,\ \Delta y_0 = 0.031,\ \Delta y_{-1} = 0.041,\ \Delta^2 y_{-1} = -0.10$

Also $h = 0.1$

$$\therefore \quad p = \frac{x - x_0}{h} = \frac{1.22 - 1.2}{0.1} = \frac{0.02}{0.1} = 0.2$$

By Stirling's formula,

$$f(x) = y_0 + p\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1}$$

$$\therefore \quad f(1.22) = 0.932 + 0.2\left(\frac{0.041 + 0.031}{2}\right) + \frac{(0.2)^2}{2}(-0.010)$$

$$= 0.932 + 0.0072 - 0.0002 = 0.939$$

Unit-IV

8. (*a*) Evaluate $\int_0^2 \frac{1}{1+x^2}\,dx$ by using Trapezoidal rule with $h = 0.25$. Also find and compare the amount of error in each case.

Sol. Here, $x_0 = 0,\ x_n = 2,\ h = 0.25$

$$\text{so} \quad n = \frac{x_n - x_0}{h} = \frac{2 - 0}{0.25} = 8$$

Also $$f(x) = \frac{1}{1+x^2}$$

Now we tabulate the function $f(x) = \frac{1}{1+x^2}$ as follows :

x :	0	0.25	0.50	0.75	1	1.25	1.50	1.75	2.0
$f(x)$:	1	0.9412	0.8	0.64	0.5	0.3902	0.3077	0.2462	0.2

By Trapezoidal rule,

$$\int_0^2 \frac{1}{1+x^2}\,dx = \frac{h}{2}[y_0 + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7) + y_8]$$

$$= \frac{0.25}{2}[1 + 2\,(0.9412 + 0.8 + 0.64 + 0.5 + 0.3902 + 0.3077 + 0.2462) + 0.2]$$

$$= 0.125\,[1 + 7.6506 + 0.2] = 1.1063.$$

Also, $$\int_0^2 \frac{1}{1+x^2}\,dx = [\tan^{-1} x]_0^2 = \tan^{-1} 2 - \tan^{-1} 0$$

$$= 1.1071$$

So, error in first case = 1.1071 – 1.1066 = 0.0005 and error in second case = 1.1071–1.1063 = 0.0008.

Clearly, the error in first case in less than that in second case. This shows that as the value of h decreases, the trapezoidal rule gives better results.

(*b*) Estimate the integral $\int_1^3 \frac{dx}{x}$ using $n = 8$ in Simpson's 1/3 rule. **(6½, 7)**

Sol. Here $n = 8,\ x_0 = 1,\ x_8 = 3$

So $$h = \frac{x_n - x_0}{n} = \frac{x_8 - x_0}{8} = \frac{3-1}{8} = 0.25$$

Now we tabulate the function $f(x) = \frac{1}{x}$ as follows :

x :	1	1.25	1.50	1.75	2.0	2.25	2.50	2.75	3
$f(x) = \frac{1}{x}$:	0	0.8	0.6667	0.5714	0.5	0.4444	0.4	0.3636	0.3333

By Simpson's 1/3 rule,

$$\int_1^3 \frac{dx}{x} = \frac{h}{3}[y_0 + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6) + y_8]$$

$$= \frac{0.25}{3}[1 + 4(0.8 + 0.5714 + 0.4444 + 0.3636) + 2(0.6667 + 0.5 + 0.4) + 0.3333]$$

$$= 1.0987$$

9. (*a*) Given $\frac{dy}{dx} = x^2 + y$, $y(0) = 1$, compute $y(0.02)$, $y(0.04)$ and $y(0.06)$ by Euler's method.

Sol. Given $f(x, y) = x^2 + y$

To find $y(0.02)$, $y(0.04)$ and $y(0.06)$ we shall carryout calculations in the following three steps with $h = 0.02$.

Step I : Taking $x_0 = 0$, $y_0 = 1$ and $h = 0.02$, we have

$$y_1 = y_0 + h\,f(x_0, y_0)$$

$$= 1 + (0.02)(0^2 + 1) = 1 + 0.02 = 1.02$$

i.e. $y(0.02) = 1.02$

Step II : Taking $x_1 = 0.02$, $y_1 = 1.02$ and $h = 0.02$, we have,

$$y_2 = y_1 + hf(x_1, y_1)$$

$$= 1.02 + (0.02)[(0.02)^2 + 1.02] = 1.0404$$

Hence $y(0.04) = 1.0404$

Step III : Taking $x_2 = 0.04$, $y_2 = 1.0404$ and $h = 0.02$, we have,

$$y_3 = y_2 + hf(x_2, y_2)$$

$$= 1.0404 + 0.02[(0.04)^2 + 1.0404] = 1.0612$$

i.e. $y(0.06) = 1.0612$

Hence $y(0.02) = 1.02$, $y(0.04) = 1.0404$ and $y(0.6) = 1.0612$

(*b*) Calculate $\int_0^4 \sqrt{64 - x^3}\, dx$ by Trapezoidal rule using 9 ordinates. (6½)

Sol. Here the whole range of integration is divided into 9 ordinates so we shall get 8 subintervals *i.e.* $n = 8$

Also $x_0 = 0$, $x_8 = 4$

So $$h = \frac{x_n - x_0}{n} = \frac{x_8 - x_0}{8} = \frac{4 - 0}{8} = 0.5$$

Now we tabulate the function $f(x) = \sqrt{64 - x^3}$ as follows :

x :	0	0.5	1	1.5	2	2.5	3	3.5	4
$f(x) = \sqrt{64 - x^3}$:	8	7.9922	7.9372	7.7862	7.4833	6.9552	6.0828	4.5962	0

By Trapezoidal rule,

$$\int_0^4 \sqrt{64 - x^3}\, dx = \frac{h}{2}[y_0 + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7) + y_8]$$

$$= \frac{0.5}{2}[8 + 2(7.9922 + 7.9372 + 7.7862 + 7.4833 + 6.9552 + 6.0828 + 4.5962) + 0]$$

$$= 0.25\,(105.6662) = 26.4166$$